Some mapping theorems for extensional dimension

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Abstract

We present some results related to theorems of Pasynkov and Torunczyk on the geometry of maps of finite dimensional compacta.

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1 Introduction

All topological spaces are assumed to be separable metrizable, $I = [0, 1]$. Recall that the covering dimension $\dim X$ is the smallest number n such that every open cover of X admits an open refinement of order $\leq n + 1$. The cohomological dimension $\dim_G X$ with respect to an abelian group G is the smallest number n such that $\check{H}^{n+1}(X, A; G) = 0$ for all closed subsets A of X.

By a classical result of Alexandroff $\dim_{\mathbb{Z}} X = \dim X$ for a finite dimensional X. Solving an outstanding problem in Dimension Theory Dranishnikov constructed in 1986 an infinite dimensional compactum (=compact metric space) of $\dim_{\mathbb{Z}} = 3$. Many classical results for the covering and the cohomological dimensions shows a great deal of similarity between the theories. Here we mention just two such results.

Theorem 1.1 *(Hurewicz [10]) Let $f : X \longrightarrow Y$ be a map of compacta. Then $\dim X \leq \dim Y + \dim f$, where $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$.*

Theorem 1.2 *(Skljarenko [1]) Let $f : X \longrightarrow Y$ be a map of compacta. Then $\dim_{\mathbb{Z}} X \leq \dim_{\mathbb{Z}} Y + \dim_{\mathbb{Z}} f$, where $\dim_{\mathbb{Z}} f = \sup\{\dim_{\mathbb{Z}} f^{-1}(y) : y \in Y\}$.*

Despite such similarities the covering dimension was investigated using mostly set-theoretic methods and the cohomological dimension was investigated using mostly algebraic methods. In general the results for the cohomological dimension were proved with difficulty exceeding similar results for the covering dimension. The proofs of the theorems mentioned above are a good illustration of the differences

in the approaches. The essentially different definitions of the dimensions seem to justify using different tools. However the following two theorems offer a common point of view on the theories.

Theorem 1.3 *(Hurewicz-Wallman [10])* $\dim X \leq n$ *if and only if any map* $f : A \to S^n$ *of a closed subset* A *of* X *into the* n*-dimensional sphere* S^n *extends over* X*.*

Theorem 1.4 *(Cohen [2])* $\dim_G X \leq n$ *if and only if any map* $f : A \to K(G, n)$ *of a closed subset* A *of* X *into an Eilenberg-MacLane complex of type* (G, n) *extends over* X*.*

Theorems 1.3 and 1.4 suggest that one may try to consider extension properties of maps into different CW-complexes hoping to create a theory encompassing both the covering and the cohomological dimension theories. Indeed, in the early 1990's Dranishnikov laid down a foundation of such theory which he called "Extension Theory". Extension Theory not only provided simpler proofs for many results in cohomological dimension theory, but it also gave an explanation of certain phenomena in cohomological dimension which are sometimes more general than expected, see [7] and [8].

Let us introduce the terminology. The extensional dimension of X does not exceed a CW-complex K, written e-dim$X \leq K$, if any map of a closed subset of X into K extends over X. Thus for the covering dimension dim$X \leq n$ if and only if e-dim$X \leq S^n$ and for the cohomological dimension dim$_G X \leq n$ if and only if e-dim$X \leq K(G, n)$. Let $f : X \longrightarrow Y$ be a map. We will use the following notation:

e-dim$f \leq K$ if e-dim$f^{-1}(y) \leq K$ for every $y \in Y$;

e-dim$Z \leq$e-dim$f$ if for every CW-complex K, e-dim$f \leq K$ implies e-dim$Z \leq K$;

e-dim$X \leq$e-dim$Y$ if for every CW-complex K, e-dim$Y \leq K$ implies e-dim$X \leq K$;

for CW-complexes K and L, $K \leq L$ if for every X, e-dim$X \leq K$ implies e-dim$X \leq L$.

This note is mainly devoted to an extensional dimension generalization of some results of Pasynkov [19] and Torunczyk [23] on the geometry of maps. For example, we obtain the following versions of Theorems 1.1 and 1.2. Note that Theorem 1.5 is an improvement of the results of [15] which provides the right estimates for cohomological dimension.

Theorem 1.5 *(cf. Remark 2, Section 5) Let* $f : X \longrightarrow Y$ *be a map of compacta and let CW-complexes* K *and* L *be such that* K *is countable,* $\mathrm{e-dim} f \leq K$ *and* $\mathrm{e-dim} Y \leq L$*. Then* $X_* = X \times I$ *can be decomposed into* $X_* = A \cup B$ *for subspaces* A *and* B *such that* $\mathrm{e-dim} A \leq K$ *and* $\mathrm{e-dim} B \leq L$*. As a result of such a decomposition* $\mathrm{e-dim} X_* \leq K * L$*.*

Theorem 1.6 *Let $f : X \longrightarrow Y$ be a map of compacta with X finite dimensional and let K and L be CW-complexes such that K is countable,* $\mathrm{e}-\dim f \leq K$ *and* $\mathrm{e}-\dim Y \leq L$*. Then* $\mathrm{e}-\dim X \leq K \wedge L$*.*

Pasynkov [19] and Torunczyk [23] proved the following remarkable theorem.

Theorem 1.7 *(Pasynkov-Torunczyk) Let $f : X \longrightarrow Y$ be a map of finite dimensional compacta X and Y. Then the following conditions are equivalent:*

(1) $\dim f \leq n$*;*

(2) there exists a σ-compact subset A of X such that $\dim A \leq n-1$ *and* $\dim f|_{X\setminus A} \leq 0$*;*

(3) almost every map $g : X \longrightarrow I^n$ has the property that the map $(f,g) : X \longrightarrow Y \times I^n$ is 0-dimensional, where almost=all but a set of first category;

(3') there exists a map $g : X \longrightarrow I^n$ such that the map $(f,g) : X \longrightarrow Y \times I^n$ is 0-dimensional.

In this note we will prove two theorems generalizing Theorem 1.7.

Theorem 1.8 *Let $f : X \longrightarrow Y$ be a map of compacta and let K be a CW-complex. Consider the following properties:*

(1) $\mathrm{e}-\dim f \leq \Sigma K$*;*

(2) there exists a σ-compact subset A of X such that $\mathrm{e}-\dim A \leq K$ *and* $\dim f|_{X\setminus A} \leq 0$*;*

(3) almost every map $g : X \longrightarrow I$ is such that for the map $(f,g) : X \longrightarrow Y \times I$, $\mathrm{e}-\dim(f,g) \leq K$*;*

(3') there exists a map $g : X \longrightarrow I$ such that for the map $(f,g) : X \longrightarrow Y \times I, \mathrm{e}-\dim(f,g) \leq K$.

Then: (3)⇒(2)⇒(1), (3)⇒(3')⇒(1) and if Y is finite-dimensional and K is countable then (1)⇒(3).

In particular all the properties are equivalent if Y is finite dimensional and K is countable.

Note that Theorem 1.7 can be derived from Theorem 1.8. Indeed, it is obvious for the properties (1) and (2) of Theorem 1.7. One can easily show by induction that the property (3) of Theorem 1.8 implies that the set of functions g satisfying the property (3) of Theorem 1.7 for $\dim f \leq n$ is dense in $C(X, I^n)$. On the other hand this set is G_δ in $C(X, I^n)$ and hence (3) holds for $\dim f \leq n$. Clearly (3)⇒(3')⇒(1) and Theorem 1.7 follows (see also Remark 1, Section 5).

It is unknown if the finite dimensional restriction on Y is necessary in Theorems 1.7 and 1.8. Some versions of these theorems without this restriction were obtained in [22, 13, 15]. The following theorem improves on these results.

Theorem 1.9 *Let $f : X \longrightarrow Y$ be a map of compacta and let K be a CW-complex. Define $f_* : X_* = X \times I \longrightarrow Y$ by $f_*(x,t) = f(x)$. Consider the following properties:*

(0) $\mathrm{e-dim} f \leq K$*;*

(1) $\mathrm{e-dim} f_* \leq \Sigma K$*;*

(2) there exists a σ*-compact subset* A *of* X_* *such that* $\mathrm{e-dim} A \leq K$ *and* $\dim f_*|_{X_* \setminus A} \leq 0$*;*

(2') a set A *satisfying (2) can be chosen such that* A *splits into compacta* $A = \cup_{i=1,\infty} A_i$ *such that each component of* A_i *admits a* 0*-dimensional map into a fiber of* f*.*

(3) almost every map $g_* : X_* \longrightarrow I$ *is such that for the map* $(f_*, g_*) : X_* \longrightarrow Y \times I$*,* $\mathrm{e-dim}(f_*, g_*) \leq K$*;*

Then (0)⇒(3)⇒(2')⇒(2)⇒(1).

Theorems 1.8 and 1.9 are proved in Section 4. Applications of Theorem 1.9 including the proofs of Theorems 1.5 and 1.6 are presented in Section 2. Section 3 is devoted to Krasinkiewicz maps which are essentially used in the proofs. In Section 5 we discuss some related results and problems.

Finally we wish to thank the referee for pointing out that Proposition 4.4 can also be derived from Lemma 2 of [23]

2 Applications

We will use the following facts.

Theorem 2.1 *([5]) Let* $f : X \longrightarrow Y$ *be a light map of compacta. Then* $\mathrm{e-dim} X \leq \mathrm{e-dim} Y$*.*

In some cases we will need a stronger result than Theorem 2.1.

Definition 2.2 *([11]) A map* $f : X \longrightarrow Y$ *of metric spaces* X *and* Y *is said to be uniformly* 0*-dimensional if there is a metric on* X *such that for every* $\epsilon > 0$ *every point in* $f(X)$ *has a neighborhood* V *in* Y *such that* $f^{-1}(V)$ *splits into a disjoint family of open sets of diam*$< \epsilon$*.*

Examples of uniformly 0-dimensional maps are: a 0-dimensional map of compacta, a 0-dimensional perfect (=closed with compact fibers) map of metric spaces, a map of compacta restricted to the union of all trivial components of its fibers.

Theorem 2.3 *([17]) Let* $f : X \longrightarrow Y$ *be a uniformly* 0*-dimensional map of metric spaces. Then* $\mathrm{e-dim} X \leq \mathrm{e-dim} Y$*.*

Theorem 2.4 *([18]) Let* K *be a countable CW-complex and let* A *be a subspace of a compactum* X *such* $\mathrm{e-dim} A \leq K$*. Then there is a completion* $A' \subset X$ *of* A *such that* $\mathrm{e-dim} A' \leq K$*.*

Theorem 2.5 *([9], see also Remark 2, Section 5) Let K and L be CW-complexes and let $X = A \cup B$ be a decomposition of a separable metric space X into subspaces A and B such that* $\mathrm{e-dim} A \le K$ *and* $\mathrm{e-dim} B \le L$. *Then* $\mathrm{e-dim} X \le K * L$.

Theorem 2.6 *([21],see also [9]) Let $X = A \cup B$ be a decomposition of a separable metric space X. Then* $\dim_{\mathbb{Z}} X \le \dim_{\mathbb{Z}} A + \dim_{\mathbb{Z}} B + 1$.

Theorem 2.7 *([7]) Let X be a compactum and let K be a simply connected CW-complex. Consider the following conditions:*

(1) $\mathrm{e-dim} X \le K$*;*

(2) $\dim_{H_i(K)} X \le i$ *for every* $i > 1$*;*

(3) $\dim_{\pi_i(K)} X \le i$ *for every* $i > 1$.

Then (2) and (3) are equivalent and (1) implies both (2) and (3). If X is finite dimensional then all the conditions are equivalent. In particular if X is finite dimensional then $\dim_G X \le n$, $n > 1$ *if and only if* $\mathrm{e-dim} X \le M(G, n)$ *where* $M(G, n)$ *is a Moore space of type* (G, n).

Theorem 2.8 *([3]) Assume that a compactum X is expressed as the union $X = A \cup B$. Then* $\dim_{\mathbb{Z}} X \le \dim_{\mathbb{Z}}(A \times B)$ *if* $\dim_{\mathbb{Z}} X^2 = 2\dim_{\mathbb{Z}} X$ *and* $\dim_{\mathbb{Z}} X \le \dim_{\mathbb{Z}}(A \times B) + 1$ *if* $\dim_{\mathbb{Z}} X^2 \neq 2\dim_{\mathbb{Z}} X$ *(that is* $\dim_{\mathbb{Z}} X^2 = 2\dim_{\mathbb{Z}} X - 1$*).*

Proposition 2.9 *([15]) Let K be a CW-complex and let each component of a compactum X be of* $\mathrm{e-dim} \le K$. *Then* $\mathrm{e-dim} X \le K$.

The following two proposition can be easily derived from the proofs of Proposition 12, [3] and Lemma 2.11, [6] respectively.

Proposition 2.10 *Let X and Y be compacta and let $A \subset X$. Then there is a completion $A' \subset X$ of A such that* $\dim_{\mathbb{Z}}(A' \times Y) = \dim_{\mathbb{Z}}(A \times Y)$.

Proposition 2.11 *Let K be a CW-complex such that* $\mathrm{e-dim}(X \times Y) \le K$ *where X and Y are non-trivial continua. Then K is simply connected.*

It was shown in [15] that for a map of compacta $f : X \longrightarrow Y$ and CW-complexes K and L, X can be decomposed into $X = A \cup B$ such that e-dim$A \le K$ and $\mathrm{e-dim} B \le L$ provided e-dim$f \le K$, e-dim$Y \le L$ and K is countable. Theorem 1.5 improves this result.

Proof of Theorem 1.5. By Theorem 1.9 there is $A \subset X_*$ such that $\mathrm{e-dim} A \le K$ and $\dim f_*|_B \le 0$ where $B = X_* \setminus A$. By Theorem 2.4 there is a completion

$A' \subset X_*$ of A with e-dim$A' \leq K$. $B' = X_* \setminus A' \subset B$ is σ-compact and hence by Theorem 2.1, e-dim$B' \leq L$. By Theorem 2.5 $\mathrm{e}-\dim X \leq K * L$. $\square$

Note that applying Theorem 2.6 to the decomposition of X_* from Theorem 1.5 for $\dim_{\mathbb{Z}}$ it follows that $\dim_{\mathbb{Z}} X + 1 = \dim_{\mathbb{Z}} X_* = \dim_{\mathbb{Z}}(A' \cup B') \leq \dim_{\mathbb{Z}} A' + \dim_{\mathbb{Z}} B' + 1$ and we have obtained Theorem 1.2, see [1] for related results.

Dranishnikov and Dydak [8] proved that if for finite dimensional compacta X and Y and CW-complexes K and L, $\mathrm{e}-\dim X \leq K$ and $\mathrm{e}-\dim Y \leq L$ then $\mathrm{e}-\dim(X \times Y) \leq K \wedge L$. Theorem 1.6 is a mapping version of Dranishnikov-Dydak's result.

Proof of Theorem 1.6. If K is disconnected then $\mathrm{e}-\dim f = 0$ and if L is disconnected then $\dim Y = 0$. These cases can be treated in a way similar to the one in the proof of Theorem 5.6 of [8] to show that $\mathrm{e}-\dim X \leq K \wedge L$.

Assume that both K and L are connected. Then $K \wedge L$ is simply connected. By Theorem 1.5, $\mathrm{e}-\dim(X_*) \leq K * L$, $X_* = X \times I$ and since $K * L$ is homotopy equivalent to $\Sigma(K \wedge L)$, $\mathrm{e}-\dim(X_*) \leq \Sigma(K \wedge L)$. Then by Theorem 2.7, $\dim_{H_{i+1}(\Sigma(K\wedge L))}(X_*) = \dim_{H_i(K\wedge L)}(X_*) = \dim_{H_i(K\wedge L)}(X) + 1 \leq i + 1$. Hence $\dim_{H_i(K\wedge L)}(X) \leq i$ and again by Theorem 2.7, $\mathrm{e}-\dim X \leq K \wedge L$. $\square$

Note that if in the proof of Theorem 1.6 one uses Theorem 5.2 (see Section 5, Remark 2 of this note) instead of Theorem 1.5 then the restriction in Theorem 1.6 that K is countable can be omitted.

The following theorem was announced by Dranishnikov in 1996. We will present a proof of this result based on Theorem 1.9.

Theorem 2.12 *(Dranishnikov) Let $f : X \longrightarrow Y$ be a map of compacta. Then:* $\dim_{\mathbb{Z}} X \leq \sup\{\dim_{\mathbb{Z}} (f^{-1}(y) \times Y) : y \in Y\}$ *if* $\dim_{\mathbb{Z}} X^2 = 2\dim_{\mathbb{Z}} X$ *and* $\dim_{\mathbb{Z}} X \leq \sup\{\dim_{\mathbb{Z}}(f^{-1}(y) \times Y) : y \in Y\} + 1$ *otherwise.*

Proof. Let $n = \sup\{\dim_{\mathbb{Z}}(f^{-1}(y) \times Y) : y \in Y\}$. Take a σ-compact $A \subset X_*$ satisfying (2') of Theorem 1.9, that is A splits into compacta $A = \cup A_i$ such that each component of each A_j admits a 0-dimensional map to a fiber of f. Then each component of $A_i \times Y$ admits a 0-dimensional map to $f^{-1}(y) \times Y$ for some $y \in Y$ and hence by Theorem 2.1 and Proposition 2.9 $\dim_{\mathbb{Z}}(A \times Y) \leq n$ and by Proposition 2.10 there is a completion $A' \subset X_*$ of A such that $\dim_{\mathbb{Z}}(A' \times Y) \leq n$. Then $B' = X_* \setminus A'$ is σ-compact and hence splits into compacta $B' = \cup B_j$ admitting 0-dimensional maps into Y. It follows that $A' \times B_j$ admits a 0-dimensional perfect map to $A' \times Y$. Thus by Theorem 2.3 $\dim_{\mathbb{Z}}(A' \times B_j) \leq n$ and hence $\dim_{\mathbb{Z}}(A' \times B') \leq n$. By Theorem 2.8 $\dim_{\mathbb{Z}} X_* \leq n + 1$ if $\dim_{\mathbb{Z}} X_*^2 = 2\dim_{\mathbb{Z}} X^*$ and $\dim_{\mathbb{Z}} X_* \leq n + 2$ otherwise. Clearly $\dim_{\mathbb{Z}} X_* = \dim_{\mathbb{Z}} X + 1$ and $\dim_{\mathbb{Z}} X_*^2 = \dim_{\mathbb{Z}} X^2 + 2$ and the theorem follows. $\square$

Note that it is unknown if the case $\dim_{\mathbb{Z}} X = \dim_{\mathbb{Z}} A + \dim_{\mathbb{Z}} B + 2$ in Theorem 2.8 ever occurs. Eliminating this case would result in eliminating the case $\dim_{\mathbb{Z}} X = \sup\{\dim_{\mathbb{Z}}(f^{-1}(y) \times Y) : y \in Y\} + 1$ in Theorem 2.12.

Our next application is to prove in a slightly stronger form the generalized Hurewicz formula obtained by Dranishnikov, Repovš and Ščepin [6].

Theorem 2.13 *(Dranishnikov-Repovš -Ščepin for a finite dimensional Y [6]) Let $f : X \longrightarrow Y$ be a map of compacta such that X is finite dimensional and Y is full-valued and let a CW-complex K be such that $\mathrm{e}-\dim(f^{-1}(y) \times Y) \leq K$ for every $y \in Y$. Then $\mathrm{e}-\dim X \leq K$.*

Proof. By Proposition 2.9 and Theorem 2.1 the theorem holds if either Y or f is 0-dimensional. If neither Y nor f is 0-dimensional then by Proposition 2.11 we may assume that K is simply connected. Fix $i > 1$ and assume $H = H_i(K) \neq 0$. Let $m = \dim_{\mathbb{Z}} Y$, $n = \dim_H f$ and $k = \sup\{\dim_H(f^{-1}(y) \times Y) : y \in Y\}$. By Theorem 2.7, $k \leq i$. Since Y is full-valued $\dim_H Y = m$ and $k = n + m$. By (2') of Theorem 1.9, Theorem 2.1 and Proposition 2.9 there exists a σ-compact $A \subset X_*$ such that $\dim_H(A) \leq n$ and $f_*|_{X_* \setminus A}$ is 0-dimensional. Let $\sigma(H)$ be the Bockstein basis for H and let $L = \vee\{M(G, n) : G \in \sigma(H)\}$ where $M(G, n)$ is a Moore space of type (G, n). Then L is countable and by Bockstein theory and Theorem 2.7 e-dim $A \leq L$. By Theorem 2.4 there is a completion $A' \subset X_*$ of A such that $\mathrm{e}-\dim A' \leq L$. Then by Theorem 2.1, $B' = X_* \setminus A'$ is of $\dim = \dim_{\mathbb{Z}} \leq m$.

Then by Theorem 2.5 $\mathrm{e}-\dim X_* = \mathrm{e}-\dim(A' \cup B') \leq L * S^m = \Sigma^{m+1} L = \vee\{M(G, n+m+1) : G \in \sigma(H)\}$. Hence by Theorem 2.7 $\dim_H X_* \leq n + m + 1$. Then $\dim_H X \leq n + m = k$. Thus $\dim_{H_i(K)} X \leq i$ and by Theorem 2.7 the theorem follows. $\square$

Note that one can avoid the use of Olszewski's completion theorem (Theorem 2.4) and Bockstein theory in the proof of Theorem 2.13 by using Theorem 5.1 instead of Theorem 2.5.

3 Krasinkiewicz maps

Krasinkiewicz [12] introduced and studied maps having the following remarkable property. We will call these maps Krasinkiewicz maps.

Definition 3.1 *([12]) A map $f : X \longrightarrow Y$ is said to be a Krasinkiewicz map if any continuum in X either contains a component of a fiber of f or is contained in a component of a fiber of f.*

Krasinkiewicz [12] showed that for a compactum X the Krasinkiewicz maps form a dense subset in $C(X, I)$. We improve this result by showing that almost every map in $C(X, I)$ is a Krasinkiewiz map.

First we present a short construction of Krasinkiewicz maps using a different approach based on [14]. Namely, we will use the following proposition actually proved in [14].

Proposition 3.2 *([14], Proposition 2.2, Proof B) Let $Z \subset X$ be a closed 0-dimensional subset of a compactum X. Then for almost every map in $C(X, I)$, Z is contained in the union of trivial components of the fibers of f.*

We construct Krasinkiewicz maps as follows. Let X be a compactum. Take a sequence of Cantor sets $C_i \subset I$ and intervals $[a_i, b_i] \subset I$, $i = 1, 2, ...$ such that $C_i \subset (a_i, b_i)$, $[a_{i+1}, b_{i+1}] \cap (C_1 \cup ... \cup C_i) = \emptyset$ and each non-empty open interval in I contains some C_i. Fix $\epsilon > 0$ and let $f \in C(X, I)$. Let $\psi_0 = f$ and assume that we have constructed $\psi_{i-1} : X \longrightarrow I$. Let $g_i : C_i \longrightarrow 2^X$ be a map onto the hyperspace 2^X of all non-empty closed subsets of X equipped with the Hausdorff topology. Define $A_i = \cup\{g_i(c) \cap \psi_{i-1}^{-1}(c) : c \in C_i\}$. Then A_i is closed in X. Let $q_i : X \longrightarrow X_i$ be the quotient map to the space X_i obtained from X by identifying each component of A_i with a singleton. Then ψ_{i-1} factors through $\phi_i : X_i \longrightarrow I$ and $Z_i = q_i(A_i)$ is 0-dimensional. By Proposition 3.2 ϕ_i can be $\epsilon/2^{i+1}$-approximated by a map $\phi_i' : X_i \longrightarrow I$ such that Z_i is contained in the union of trivial components of the fibers of ϕ_i'. Let $\psi_i = \phi_i' \circ q_i : X \longrightarrow I$. Then ψ_i is $\epsilon/2^{i+1}$-close to ψ_{i-1} and it easy to see that we may assume that $\psi_i(x) = \psi_{i-1}(x)$ if $\psi_{i-1}(x) \in I \setminus (a_i, b_i)$. Then $\psi = \lim \psi_i$ is ϵ-close to $f = \psi_0$ and one can easily observe that ψ is a Krasinkiewicz map. Moreover, ψ satisfies the following condition:

(*) for every continuum $F \subset X$ such that $\psi(F)$ is not a singleton there is a subset $D \subset \psi(F)$ dense in $\psi(F)$ such that for every $d \in D$,$\psi^{-1}(d) \cap F$ is the union of components of $\psi^{-1}(d)$.

Proposition 3.3 *Let X be a compactum. The set of maps in $C(X, I)$ satisfying the condition (*) is a dense G_δ-subset of $C(X, I)$.*

Proof. Let H_* =the set of maps in $C(X, I)$ satisfying (*). We have already shown that H_* is dense in $C(X, I)$. For $0 \leq p < q \leq 1$ denote by $H(p, q, n)$ the set of functions $f \in C(X, I)$ such that there is a continuum $F \subset X$ such that $[p, q] \subset f(F)$ and for every $d \in [p, q]$ there is a component of the fiber $f^{-1}(d)$ intersecting F and containing a point x such that $\text{dist}(x, F) \geq 1/n$. It is easy to check that $H(p, q, n)$ is closed in $C(X, I)$. Denote $H = \cup\{H(p, q, n) : p, q \text{ are rationals}, 0 \leq p < q \leq 1, n = 1, 2, ..\}$

Let us show that $H_* = C(X, I) \setminus H$. It is clear that $H_* \subset C(X, I) \setminus H$. If ψ does not satisfy (*) then there are a continuum $F \subset X$ and an interval $[a, b] \subset \psi(F)$ such that for every $d \in [a, b]$ there is a component of $\psi^{-1}(d)$ intersecting both F and $X \setminus F$. Let $A_n = \{d \in [a, b] :$ there are a component C of $\psi^{-1}(d)$ and a point $x \in C$ such that $C \cap F \neq \emptyset$ and $\text{dist}(x, F) \geq 1/n\}$. Then A_n is closed, $[a, b] = \cup A_n$ and hence there are n and a non-degenerate interval $[p, q]$ with rational endpoints such

that $[p,q] \subset A_n$. Then $\psi \in H(p,q,n)$ and we proved that $H_* = C(X,I) \setminus H$. Thus H_* is G_δ in $C(X,I)$ and the proposition follows. $\square$

Clearly each map satisfying (*) is a Krasinkiewicz map and Proposition 3.3 implies:

Theorem 3.4 *Let X be a compactum. Almost every map in $C(X,I)$ is a Krasinkiewicz map.*

Proposition 3.5 *Let $f : X \longrightarrow Y$ be a Krasinkiewicz map of compacta X and Y. Denote $C(f)$ =the union of all non-trivial components of the fibers of f. Then there are compacta $A_1, A_2, ... \subset X$ such that $C(f) = \cup A_i$ and each component of A_i is contained in a fiber of f. In particular* $\mathrm{e-dim} C(f) \leq \mathrm{e-dim} f$.

Proof. Let C_n = the union of all non-trivial components of f of diam$\geq 1/n$. Clearly C_n is closed and $C(f) = \cup_{n=1,\infty} C_n$. Cover C_n by finitely many closed subsets $B^n_1, B^n_2, ... \subset C_n$ of diam$< 1/n$. Then a component of B^n_j cannot contain a component of a fiber of f and hence must be contained in a componet of a fiber of f. Enumerate $\{B^n_j\}$ as $\{A_1, A_2, ...\}$ and we are done. $\square$

A continuum X is said to be hereditarily indecomposable if for every pair of intersecting subcontinua $A, B \subset X$, either $A \subset B$ or $B \subset A$. A map f is said to be a Bing map if each component of each fiber of f is hereditarily indecomposable.

Theorem 3.6 *([13]) Let X be a compactum. Almost every map in $C(X,I)$ is a Bing map.*

By a Bing-Krasinkiewicz map we mean a map which is both Bing and Krasinkiewicz. By Theorems 3.4 and 3.6 for a compactum X almost every map in $C(X,I)$ is Bing-Krasinkiewicz. It is easy to verify:

Proposition 3.7 *Let $f : X \longrightarrow Y$ be a Bing-Krasinkiewicz map of compacta. Then for every map $g : X \longrightarrow Z$ to a compactum Z the map $(f,g) : X \longrightarrow Y \times Z$ is also Bing-Krasinkiewicz.*

4 Proofs of Theorems 1.8 and 1.9

We first present some facts used in the proofs.

Proposition 4.1 *([22]) Let Y be a compactum. Then there is a σ-compact 0-dimensional subset $A \subset Y \times I$ such that for the projection $p : Y \times I \longrightarrow Y$,* $\dim p|_{(Y\times I)\setminus A} = 0$.

Such a subset A is constructed as follows. Let $C_1, C_2, ... \subset I$ be a sequence of Cantor sets such that each non-empty open subset of I contains some C_i. Let a map $f_i : C_i \longrightarrow Y$ be onto. Define $A_i = \{(f_i(c), c) : c \in C_i\}$. Then $\dim A_i = 0$ and $A = \cup A_i$ satisfies Proposition 4.1.

Proposition 4.2 *Let X be a compactum, let K be a CW-complex and let a map $f : X \longrightarrow I$ be of $\mathrm{e-dim} \leq K$. Then $\mathrm{e-dim} X \leq \Sigma K$.*

Proof. Let $g : F \longrightarrow \Sigma K$ be a map of a closed subset F of X. Since $\mathrm{e-dim} f \leq K \leq \Sigma K$, g can be extended over a small neighbourhood of each fiber of f and hence there is a partition $0 = t_1 < t_2 < ... < t_k = 1$ of I such that g extends to $g_i : F_i = F \cup f^{-1}([t_i, t_{i+1}]) \longrightarrow \Sigma K$ for every $i = 1, k-1$. Since $\mathrm{e-dim} f \leq K$, $\mathrm{e-dim}(f^{-1}(t) \times I) \leq \Sigma K$ for every $t \in I$ and hence $g_i|_{f^{-1}(t_{i+1})}$ and $g_{i+1}|_{f^{-1}(t_{i+1})}$ are homotopic for every $i = 1, k-2$. It follows that there is a map $g' : X \longrightarrow \Sigma K$ which is homotopic on F_i to g_i for each $i = 1, k-1$ and hence g can be extended over X. $\square$

Theorem 4.3 *([4]) Let K and L be countable CW-complexes and let for a compactum X, $\mathrm{e-dim} X \leq K * L$. Then there is a σ-compact $A \subset X$ such that $\mathrm{e-dim} A \leq K$ and $\mathrm{e-dim}(X \setminus A) \leq L$.*

Proposition 4.4 *Let $f : X \longrightarrow Y$ be a light map of finite dimensional compacta X and Y. Then almost every map $g : X \longrightarrow I$ has the property that each fiber of the map $(f, g) : X \longrightarrow Y \times I$ contains at most $\dim Y + 1$ points;*

Proof. Let $\delta > 0$ and consider the set $\mathcal{G}$ of maps $g : X \longrightarrow I$ having the property that each fiber of (f, g) contains at most $k+1$ points having pairwise distances $\geq \delta$. Clearly $\mathcal{G}$ is an open subset of the mapping space $C(X, I)$. We will show that $\mathcal{G}$ is also dense proving the proposition.

Take a map $g' : X \longrightarrow I$ and fix $\epsilon > 0$. Let $\phi_\alpha : Y \longrightarrow I$ be a finite set of maps forming a partition of unity for Y such that $V_\alpha = \mathrm{supp}\phi_\alpha = \phi_\alpha^{-1}(0, 1]$ is a cover of order $k + 1$. Since $\dim f = 0$ we may assume that each $f^{-1}(V_\alpha)$ splits into a finite family of disjoint open sets $U^1_\alpha, U^2_\alpha, ...$ such that diam $U^i_\alpha \leq \delta/3$ and $\mathrm{diam} g'(U^i_\alpha) \leq \epsilon/3$ for every i. Let $\psi^i_\alpha : X \longrightarrow I$ be defined by $\psi^i_\alpha(x) = \phi_\alpha(f(x))$ if $x \in U^i_\alpha$ and $\psi^i_\alpha(x) = 0$ if $x \in X \setminus U^i_\alpha$. Then ψ^i_α form a partition of unity for X. Choose real numbers $0 < b^i_\alpha < 1$ such that $\mathrm{dist}(b^i_\alpha, g'(U^i_\alpha)) < \epsilon/6$ and b^i_α have the following general position property: for every l any collection of $l+1$ points in R^l with the coordinates formed from different elements b^i_α is not contained in a hyperplane of R^l.

Define $g : X \longrightarrow I$ by $g(x) = \Sigma_{\alpha,i}\psi^i_\alpha(x)b^i_\alpha$. Then $|g(x) - g'(x)| \leq \epsilon$. We will show that $g \in \mathcal{G}$. Let $a \in Y$, $b \in I$ and let $x_1, x_2, ... \in X$ be such that $\mathrm{dist}(x_{j_1}, x_{j_2}) \geq \delta, j_1 \neq j_2$ and $(f, g)(x_j) = (a, b)$. Let $V_{\alpha_1}, ..., V_{\alpha_l}, l \leq k+1$ be the

sets containing a. Then $x_j \in U^{i(j,t)}_{\alpha_t}$ and $i(j_1,t) \neq i(j_2,t)$ if $j_1 \neq j_2$ for every $t = 1,l$. Now $g(x_j) = \Sigma_{t=1,l}\psi^{i(j,t)}_{\alpha_t}(x_j)b^{i(j,t)}_{\alpha_t} = \Sigma_{t=1,l}\phi_{\alpha_t}(f(x_j))b^{i(j,t)}_{\alpha_t} = \Sigma_t\phi_{\alpha_t}(a)b^{i(j,t)}_{\alpha_t} = b$. Then for every j, $(b^{i(j,1)}_{\alpha_1}, b^{i(j,2)}_{\alpha_2}, ..., b^{i(j,l)}_{\alpha_l})$ is a solution of the linear equation $\Sigma_t\phi_{\alpha_t}(a)s_t = b$ for the variables s_t. Since $\Sigma_t\phi_{\alpha_t}(a) = 1$ there is at least one non-zero coefficient in the equation and according to our choice of b^i_α there are at most l solutions of this equation of the form $(b^{i(j,1)}_{\alpha_1}, b^{i(j,2)}_{\alpha_2}, ..., b^{i(j,l)}_{\alpha_l})$ and hence $j \leq l \leq k+1$. Thus we have proved that $g \in \mathcal{G}$ and the proposition follows. □

Proof of Theorem 1.8.

(1)⇒(3) Take a finite-to-one map $\psi : C \longrightarrow Y$ from a Cantor set C onto Y and let $Z = \{(c,x) \in C \times X : \psi(c) = f(x)\}$ be the pullback of ψ and f with the projections $p_C : Z \longrightarrow C$ and $p_X : Z \longrightarrow X$. Then $\mathrm{e-dim} p_C = \mathrm{e-dim} f \leq \Sigma K$ and by Proposition 2.9 $\mathrm{e-dim} Z \leq \Sigma K$. By Theorem 4.3 there are 0-dimensional compact subsets $A_1, A_2, ...$ of Z such that $\mathrm{e-dim}(Z \setminus A) \leq K$ where $A = \cup A_i$. Then $\mathrm{e-dim}(f^{-1}(y) \setminus p_X(A)) \leq K$ for every $y \in Y$. Since ψ is finite-to-one we have that $f|_{p_X(A_i)}$ is 0-dimensional for every i. Hence by Proposition 4.4 almost every map $g : X \longrightarrow I$ is such that (f,g) is at most $(\dim Y + 1)$-to-1 on $p_X(A)$. Let $(y,t) \in Y \times I$. Then $(f,g)^{-1}(y,t) \subset (f^{-1}(y) \setminus p_X(A)) \cup ((f,g)^{-1}(y,t) \cap p_X(A))$ and hence $\mathrm{e-dim}(f,g) \leq K$. □

(3)⇒(2) By Theorems 3.4 and 3.6 there is a Bing-Krasinkiewcz map $g : X \longrightarrow I$ such that $\mathrm{e-dim}(f,g) \leq K$. By proposition 4.1 there is a σ-compact 0-dimensional set $S \subset Y \times I$ such that $\dim((\{y\} \times I) \setminus S) = 0$ for every $y \in Y$. Then by Proposition 2.9, $A' = (f,g)^{-1}(S)$ is of $\mathrm{e-dim} \leq K$. By Proposition 3.7 (f,g) is Bing-Krasinkiewicz and by Proposition 3.5, $A'' = C((f,g))$ is of $\mathrm{e-dim} \leq K$. Then $A = A' \cup A''$ is σ-compact and we show that $f|_{X \setminus A}$ is 0-dimensional. Let $h_m : X \longrightarrow Z$ and $h_l : Z \longrightarrow Y \times I$ be the monotone-light decomposition of $(f,g) = h_l \circ h_m$ with h_m monotone and h_l light. Then for every $y \in Y$, $h_m|_{f^{-1}(y)\setminus A''}$ is a homeomorphism and $h_m(f^{-1}(y)\setminus A') \subset h_l^{-1}((\{y\}\times I)\setminus S)$. Since $h_l^{-1}((\{y\}\times I)\setminus S)$ is 0-dimensional and $h_m(f^{-1}(y) \setminus A) \subset h_l^{-1}((\{y\} \times I) \setminus S)$, $f^{-1}(y) \setminus A$ is also 0-dimensional and therefore $\dim f|_{X\setminus A} = 0$. □

(3)⇒(3') is obvious. □

(3')⇒(1) by Proposition 4.2. □

(2)⇒(1) by Theorem 2.5. □

Proof of Theorem 1.9.

(0)⇒(3) Let $g_* : X_* \longrightarrow I$ be a Bing map. Then a fiber $(f_*, g_*)^{-1}(y,t) = g_*^{-1}(t) \cap (f^{-1}(y) \times I), y \in Y, t \in I$ contains no non-degenerate interval and hence the projection of $(f_*, g_*)^{-1}(y,t)$ onto $f^{-1}(y)$ is a 0-dimensional map. Thus by Theorem 2.1 $\mathrm{e-dim}(f_*, g_*)^{-1}(y,t) \leq \mathrm{e-dim} f^{-1}(y)$ and hence $\mathrm{e-dim}(f_*, g_*) \leq K$. Note that by Theorem 3.6 almost every map is a Bing map and we are done. □

(3)⇒(2') Replace X, f and g by X_*, f_* and g_* respectively and use the construction and notation of the proof (3)⇒(2) of Theorem 1.8. We only need to check that A can be decomposed into a countable family of compacta whose components admit 0-dimensional maps into the fibers of f. Each continuum contained in A' is contained

in a fiber of (f_*, g_*). By Proposition 3.5 A'' can be decomposed into a countable family of compacta whose components are contained in the fibers of (f_*, g_*). Each continuum in a fiber of (f_*, g_*) is hereditarily indecomposible and hence its projection to the corresponding fiber of f is a 0-dimensional map. $\square$

(2')⇒(2) is obvious. $\square$

(2)⇒(1) follows from Theorem 2.5. $\square$

5 Remarks

Remark 1. The properties (1),(2) and (3) of Theorem 1.7 are equialent to:

(4) (cf. [20]) *almost every map* $g : X \longrightarrow I^{n+1}$ *has the property that each* $fiber$ of *the map* $(f, g) : X \longrightarrow Y \times I^{n+1}$ *contains at most* $\dim Y + n + 1$ *points.*

Indeed, by Proposition 4.4 and Theorem 1.7 this property holds if $\dim f \leq n$. Assume that there is $y \in Y$ such that $\dim f^{-1}(y) \geq n + 1$. Take disjoint closed subsets $F_1, F_2, ..., F_k \subset f^{-1}(y)$, $k = \dim Y + n + 2$ such that $\dim F_i \geq n + 1$ and let $g : X \longrightarrow I^{n+1}$ be such that $g|_{F_i}$ is essential for each i (that is $g|_{g^{-1}(\partial I^{n+1}) \cap F_i}$ cannot be extended over F_i as a map to ∂I^{n+1}). Then any sufficiently close approximation of g must be at least $(\dim Y + n + 2)$-to-1 on $f^{-1}(y)$ and this contradiction shows that (4) implies that $\dim f \leq n$.

Remark 2. The proof of Theorem 2.5 given in [9] applies to prove the following stronger result.

Theorem 5.1 *Let K and L be CW-complexes and let a separable metric space X be decomposed into subsets $X = A \cup B$ such that* $\mathrm{e-dim} A \leq K$ *and for every subset F closed in X and contained in B,* $\mathrm{e-dim} F \leq L$. *Then* $\mathrm{e-dim} X \leq K * L$.

Theorem 5.1 allows one to avoid the use of Olszewski's completion theorem in Theorem 1.5 and to extend it to the following conclusion omitting the requirement that K is countable.

Theorem 5.2 *Let K and L be CW-complexes such that for a map of compacta* $f : X \longrightarrow Y$, $\mathrm{e-dim} f \leq K$ *and* $\mathrm{e-dim} Y \leq L$. *Then* $\mathrm{e-dim} X_* \leq K * L$ *where* $X_* = X \times I$.

E. Ščepin conjectured that $\mathrm{e-dim}(X_1 \cup X_2) \leq \mathrm{e-dim}(X_1 * X_2)$. This would significantly improve Theorem 2.5. Let us state without a proof the following result related to Ščepin's conjecture.

Theorem 5.3 *Let* $X = X_1 \cup X_2$ *be a decomposition of a compactum X. Then* $\mathrm{e-dim} X \leq \mathrm{e-dim}(\beta X_1 * \beta X_2)$, *where βX is the Stone-Cech compactification of X.*

Theorem 5.3 implies, for example, that $\dim X \le \dim(\beta X_1 \times \beta X_2) + 1$. Note that Theorem 5.3 does not seem to be useful for infinite dimensional spaces, see [16].

Remark 3. It seems to be of interest to know some dimensional properties of $B = X_* \setminus A$ in Theorem 1.9, especially, when K is uncountable and it is unknown if A has a completion of the same extensional dimension. In general, Theorem 2.1 does not hold for spaces which are not σ-compact and hence the 0-dimensionality of $f_*|_B$ does not give much information about B. In view of Theorem 2.3 one is tempted to replace the 0-dimensionality by the uniform 0-dimensionality.

Unfortunately Theorems 1.7, 1.8 and 1.9 do not hold if the 0-dimensionality is replaced by the uniform 0-dimensionality. Indeed, let $p : I^n \longrightarrow I$, $n \ge 2$ be the projection $p(x_1, ..., x_n) = x_n$. Then for every σ-compact $(n-2)$-dimensional subset A of I^n and every interval (a, b) in I, $p^{-1}((a, b)) \setminus A$ is connected.

However one can observe that the following result is contained in the proof of Theorem 1.9.

Theorem 5.4 *Let $f : X \longrightarrow Y$ be a map of compacta and let $f_* : X_* = X \times I \longrightarrow Y$ be defined by $f_*(x, t) = f(x)$. Then for almost every map $g_* : X_* = X \times I \longrightarrow I$ there is a σ-compact $A \subset X_*$ such that A splits into a countable family of compacta whose components admit 0-dimensional maps into the fibers of f and for $B = X_* \setminus A$, $f_*|_B$ is 0-dimensional and $(f_*, g_*)|_B$ is uniformly 0-dimensional.*

In particular $\mathrm{e-dim} A \le \mathrm{e-dim} f$ and $\mathrm{e-dim} B \le \mathrm{e-dim} \Sigma Y$.

Remark 4. Let us repeat that it is unknown if Theorems 1.7 and 1.8 hold without any dimensional restriction on Y. We end this note with posing the following problem.

Problem 5.5 *(cf. Theorem 4.3) Let $f : X \longrightarrow Y$ be a map of compacta with Y finite dimensional and let K and L be countable CW-complexes such that $\mathrm{e-dim} f \le K * L$. Does there exist a σ-compact $A \subset X$ such that $\mathrm{e-dim} A \le K$ and $\mathrm{e-dim} f|_{X \setminus A} \le L$?*

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